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The Index Theorem of topological regular variation and its applications

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ABSTRACT

We develop further the topological theory of regular variation of [N.H. Bingham, A.J. Ostaszewski, Topological regular variation: I. Slow variation, LSE-CDAM-2008-11]. There we established the uniform convergence theorem (UCT) in the setting of topological dynamics (i.e. with a group T acting on a homogenous space X), thereby unifying and extending the multivariate regular variation literature. Here, working with real-time topological flows on homogeneous spaces, we identify an index of regular variation, which in a normed-vector space context may be specified using the Riesz representation theorem, and in a locally compact group setting may be connected with Haar measure.

1. Introduction

The theory of regular variation as originally developed is a theory of positive functions of a positive variable, and so belongs to Real Analysis; for a full treatment, see e.g. [7]. In the classical theory, one assumes measurability, but can also handle the topological case, where one assumes instead the property of Baire, in parallel. More recently, much effort has been devoted to building up a theory in higher dimensions (finitely or infinitely many), motivated principally by probability theory. In addition, recent work by the authors has succeeded in solving the main foundational problem of the classical theory – finding a common generalisation of measurability and the Baire property. The resulting theory may be called combinatorial, as the techniques used belong to infinite combinatorics. However, it emerges that it is the topological, rather than the measurable, case that is the more important (see e.g. [10–12]), and so we call the resulting theory that of topological regular variation.

In passing from slow variation to regular variation, the key concept is that of the index of regular variation. Our object here is to develop the theory needed to extend the index of regular variation to the full generality of the new theory of topological regular variation. We call our main result the Index Theorem of the title.

In Section 2 below we set up the necessary preliminaries. In Section 3 we formulate, prove and discuss the Index Theorem.

2. Preliminaries

Recall that in the classical (Karamata) setting of regular variation $f:\mathbb{R}_+\to\mathbb{R}_+$ is *regularly varying* if

$$f(\lambda x)/f(x)\to g(\lambda)\in\mathbb{R}_+\quad (x\to\infty)\ \forall\lambda>0.$$

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The main result, the uniform convergence theorem (UCT), is that, subject to appropriate regularity on f, the convergence is uniform on compact sets of λ in $\mathbb{R}_+$. For a textbook treatment, background and references, see e.g. [7].

The Uniform Convergence Theorem (UCT) in a topological dynamics setting is established in [10] so provides the foundations for a topological theory of regular variation. Let X be a *phase* space, a homogeneous metric space, specifically a group with identity e_X. If a metrizable group T acts on the space X by mapping (t,x) continuously to $t(x)$, then we say that T is an *action* space for X; we treat it as a subgroup of $\mathrm{Auth}(X)$, the group of auto-homeomorphisms of X (this follows the notation of [6]). We say that $x \to tx$ is *bounded* if $\|t\|_T = d_T(t, e_T)$ is finite, where $d_T(t,t')$ denotes the supremum metric $\sup_x d_X(t(x), t'(x))$. We restrict T to be a subgroup of $\mathcal{H}(X)$, the group of bounded elements of $\mathrm{Auth}(X)$, with supremum metric. We say that $h : X \to \mathbb{R}$ is *regularly varying on the action space* if $\partial_X h(t) := \lim h(tx_n)h(x_n)^{-1}$ exists for every *divergent* sequence $\{x_n\}$ (with $\|x_n\|_X := d(x_n, e_X) \to \infty$). Also we say that $h : X \to \mathbb{R}$ is *regularly varying in the phase space* if $\partial_T h(x) := \lim h(tx)h(te_X)^{-1}$ exists for every *divergent* sequence of homeomorphisms t in T. Here divergent may be taken either in the uniform sense that $\|t\|_T \to \infty$, or in the pointwise sense that, for each x, $d(t(x), e_X) \to \infty$. Then the *Primal and Dual UCT* assert that each of the two limit functions $\partial h(.)$, on the state or action space, is a homomorphism and convergence to the limit is uniform on compact sets (for *h Baire*, but a theorem of Kodaira in [36] permits the substitution of *measurable* in the sense of Haar measure – see below; for further details see [9], Section 5). It is this duality which marks out the topological dynamics theory from the group-theoretic approach advanced by Bajšanski and Karamata [1]. Actually, however, the essential difference between them reduces to the question of what divergence structures each theory admits. (This is shown in [43], through an intermediary, an inner direct product construction, which reduces flows to multiplicative actions by subgroups.) Notwithstanding, the topological theory yields direct and immediate interpretations of current uses of regular variation.

In the present paper we develop further the topological dynamics theory with the goal of identifying an index of variation. When flows are directed by an additive group, for instance by the real line $\mathbb{R}$, and so interpreted as providing a notion of direction, the homomorphism theorem takes on a sharper form, leading to a representation theory for the limit function ∂h of a regularly varying function h, the basis of which is the spectral theorem. In this respect we go beyond the Euclidean case established in the Bajšanski–Karamata theory [1] and the Meerschaert–Scheffler theory [39] for invertible matrices, i.e. in $GL(\mathbb{R}, n)$. We refer to $\mathbb{R}$ in its capacity to direct flows as the *time domain* and the associated flows as *real time flows*.

The theory established in our first paper [10] is concerned with slowly varying functions. This is further developed in two companion papers. In [11] we investigate the Fundamental Theorems of regular variation (UCT, Characterization and Representation Theorems), in particular the UCT in the form of Goldie's Bounded Equivalence Theorem in order to clarify its standing in relation to [10]; Goldie's theorem implicitly uses flows in discrete time in considering the limit $\lim_n h(e^t e^n)/h(e^n)$. In [12] we are concerned with regularly varying functions (e.g., we show these obey the chain rule, and in the non-commutative context we characterize pairs of regularly varying functions whose product is regularly varying. The latter requires the use of a 'differential modulus' akin to the modulus of Haar integration.)

Taken together our five papers, this and [10–12,43], firmly and fully establish the foundations for an extension of the classical theory rich enough to capture all of its modern applications both in $\mathbb{R}^d$ and in classical function spaces (cf. [2,4,15, 33,42,47–49]).

There is a natural connection between regular variation as a branch of analysis and as a branch of the part of algebra concerned with topological dynamics, although the focus of research is inverted as we now explain (see particularly [21]). For T a topological group, X a topological space and a fixed flow $\varphi : T \times X \to X$, context usually allows the notation $\varphi(t,x)$ to be abbreviated in algebraic fashion to tx (or xt as is preferred in [20]). Then a continuous $\sigma : T \times X \to K$, with K a topological group, is said to be a *cocycle* if

$$\sigma(st,x) = \sigma(t,x)\sigma(s,tx).$$

(Compare [43].) In the multiplicative formulation of classical regular variation (i.e., with $X = T = K = \mathbb{R}_+^* = \mathbb{R}_+ \backslash \{0\}$), for continuous $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$, let us put

$$\sigma_f(t,x) := f(tx)f(x)^{-1}.$$

Then σ_f is a cocycle, with range (= co-domain) the group $K = \mathbb{R}_+^*$, since

$$\frac{f(stx)}{f(x)} = \frac{f(tx)}{f(x)}\frac{f(stx)}{f(tx)}.$$

Thus f is regularly varying if there is a function $g : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that, for each t,

$$\sigma_f(t,x) \to g(t), \quad \text{as } x \to \infty.$$

In the context of topological dynamics the study of cocycles σ may be reduced to those of the form $\sigma : T \times T \to K$; by embedding T in a larger compact group, limit objects such as $g(t)$ may then be identified with $\sigma_f(t,p)$ for p a limit point of T outside T. That is, the limiting function of regular variation is just a section of a cocycle; however, it is all of the cocycle, not its section, that is of interest in topological dynamics, and one may ask whether a cocycle σ is a *coboundary*, that is, whether for some continuous $f : X \to K$

$$f(tx) = f(x)\sigma(t,x),$$

so that σ is then the coboundary of f. When K is a compact group, in certain situations (X, T) has a 'minimal' extension on which the answer is affirmative and the solution is the naturally associated function $f_\sigma(t) := \sigma(t, e)$, made unique by the condition $f_\sigma(e) = e$ – see [20]. (The natural setting is T discrete, as the choice $T = \mathbb{Z}$ is consistent with the classical context where one may w.l.o.g. take limits over x running through $\mathbb{Z}$; then T is embedded in βT, the Stone–Čech compactification, and this leads to taking the minimal extension M to be a fixed minimal right ideal in βT; these notions are defined in [20]. A critical part here is played by the assured existence of an idempotent u in M and by subgroups of the group $G = Mu$.)

Definitions 1. Let X be a homogeneous metric space with distinguished points z_0 and s_0 (e.g. a metrizable topological group). A continuous function $\varphi : \mathbb{R} \times X \to X$ is said to be a (*real*) flow, the only kind to be considered here, if

$$\varphi\bigl(t, \varphi(\tau, x)\bigr) = \varphi(t + \tau, x)$$

and $\varphi(0, x) = x$. (See e.g. [28] for the general theory, or the more recent [20]; [5], especially Ch. 1, is dedicated to real flows.)

Thus each of the functions $\varphi^t(x) := \varphi(t, x)$ is a homeomorphism of X, having as inverse $\varphi^{-t}(x)$. Moreover, $\mathcal{G}(\varphi) := \{\varphi^t\colon\ t \in \mathbb{R}\}$ is a subgroup of $\mathrm{Auth}(X)$, the group of all auto-homeomorphisms of X. We refer to it as the *transformation group* of the flow. For any subgroup Φ of $\mathrm{Auth}(X)$, we will say that the flow φ is a *Φ-flow* if $\mathcal{G}(\varphi)$ is a subgroup of Φ.

We denote by $\mathcal{O}(x)$ the *orbit* $\{\varphi^t(x) : t \in \mathbb{R}\}$.

We say that φ is a *monotone divergent flow* with *source* s_0 if the following properties holds:

(i) convergence to the source:

$$d\bigl(\varphi(t, x), s_0\bigr) \to 0, \quad \text{as } t \to -\infty,$$

(ii) the homeomorphisms φ^t are (monotonically) *divergent to infinity*, namely, for each x

$$d\bigl(\varphi(t, x), x\bigr) \to \infty \text{ monotonically}, \quad \text{as } t \to +\infty,$$

which corresponds to the simple divergence notion introduced earlier.

Remark. The latter property implies that $\{\varphi^t(.)\}$ is a divergent sequence on each of the orbits regarded as subspaces of X. It also guarantees the crimping property on each of these. We recall from [10] the definition that (X, d) is *locally $\mathcal{H}$-crimping*, or simply has the *crimping property*, if for any $a \in X$ and any sufficiently small $\varepsilon > 0$, there is $\delta > 0$ such that for all b with $d(a, b) < \delta$ there exists $h \in \mathcal{H}(X)$ with $\|h\| < \varepsilon$ and $b = h(a)$. This is a form of strong local homogeneity, as defined by [24] (see [59,10,57]).

Lemma 1. *Each orbit regarded as a subspace has the crimping property.*

Proof. For $x_n \to x_0$ a sequence in the orbit of x_0, suppose that $x_n = \varphi(t_n, x_0)$ for some t_n. Then $t_n \to 0$. Indeed, suppose w.l.o.g. that $\tau = \inf t_n > 0$; then, for $t \geqslant \tau$, we have by monotonicity

$$d\bigl(\varphi(t, x_0), x_0\bigr) \geqslant d\bigl(\varphi(\tau, x_0), x_0\bigr) > 0,$$

and so $d(x_n, x_0) = d(\varphi(t_n, x_0), x_0) \geqslant d(\varphi(\tau, x_0), x_0) > 0$, which contradicts $d(x_n, x_0) \to 0$. Now put

$$\psi_n(x) := \varphi(t_n, x);$$

then $\psi_n(x) \to x$ for each x. In general this will be uniform convergence on compact sets of x. $\square$

Examples. (i) *Radial flow* $\varphi(t, x) = tx := (tx_1, tx_2)$ in the plane has source 0. In the extreme-value literature, for which see [47–49], the plane is replaced by a cone with vertex 0.

(ii) *Shift flow* in the plane, in direction e, namely $\varphi_e(t, x) := x + te$. Two special cases arise when we take the direction to be one of the natural base vectors, $e = e_i$. These yield horizontal and vertical flows. Under a logarithmic transformation $(t, x) \to (\log t, (\log x_1, \log x_2))$, the punctured quadrant $\mathbb{R}^2_+\backslash\{0\}$ is homeomorphic to the plane $\mathbb{R}^2$ and the radial flow tx of (i) is represented by $\varphi_{(1,1)}(t, x) = (x_1 + t, x_2 + t)$. The source is now at minus infinity.

(iii) *Direct sum flow* in the positive quadrant $\mathbb{R}^2_+$. Let $r(t)$ and $s(t)$ be strictly increasing continuous functions from $\mathbb{R}$ to $\mathbb{R}$. The flow

$$\varphi(t, x) := \bigl(r(t)x_1, s(t)x_2\bigr)$$

corresponds to the definition of regular variation of [15], Section 2. It is the sum of the flows $x \to (r(t)x_1, x_2)$ and $x \to (x_1, s(t)x_2)$ in the sense defined later.

The case $r(t) = t^\alpha$, $s(t) = t^\beta$, say with $\beta = \lambda\alpha$, may be simplified in two steps. First, by re-parametrization the flow may be replaced by

$$\varphi(r, x) := \bigl(rx_1, r^\lambda x_2\bigr).$$

This flow may in turn be transformed by conjugacy to a radial flow, as follows. Define the homeomorphism $\eta(x_1, x_2) = (x_1, x_2^\lambda)$. Thus

$$\varphi\big(r, \eta(x)\big) = \big(rx_1, r^\lambda x_2^\lambda\big) = \big(rx_1, (rx_2)^\lambda\big),$$

and so

$$\varphi_H(r, x) := \eta^{-1}\big(\varphi\big(r, \eta(x)\big)\big) = \eta^{-1}\big(rx_1, (rx_2)^\lambda\big) = (rx_1, rx_2).$$

Remark. The choice of such a flow is dictated by the need to simplify the function $x^\alpha y^\beta$.

(iv) *Vector flows*. For a vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$ that is Lipschitz continuous, there exists locally at $t = 0$ a unique solution $\varphi(t, x) = \varphi_V(t, x)$ of the equation $\dot\varphi(t, x) = V(\varphi(t, x))$ with initial condition $\varphi(0, x) = x$. For the linear field $V(x) = Ax$, the flow is given globally by $\varphi_A(t, x) := \exp(At)x$. Linear fields, and thereby their flows, may be added; for future reference note that if A, B commute then $\varphi_{A+B}(t, x) = \exp(At)\exp(Bt)x = \varphi_A(t, \varphi_B(t, x))$ and, since $\dot\varphi_A(t, x) = A\varphi_A(t, x)$, we then have

$$\dot\varphi_{A+B}(t, x) = (A + B)\varphi_{A+B}(t, x) = \dot\varphi_A\big(t, \varphi_B(t, x)\big) + \dot\varphi_B\big(t, \varphi_A(t, x)\big).$$

For general A the orbits may be represented by combinations of exponential functions $e^{\lambda t}$ corresponding to eigenvalues of A multiplied by polynomials in t, i.e. factors that are slowly varying (in a relative sense). Thus without loss of generality one may as well assume that $A = \Lambda$ a diagonal matrix; in this case writing e^{x_i} for x_i we obtain the additive formulation $\varphi_A(t, x) := (\exp(\lambda_i t + x_i))$. That is, under a canonical transformation of the underlying space, we obtain an affine flow $\varphi_A(t, x) := (\lambda_i t + x_i)$. See later for consequences of this.

(v) *Semi-flows*. Identifying the Borel sets of $\mathbb{R}^d$ as the subspace of corresponding indicator functions in L^1 one may consider intersection as defining a semi-group directed flow $(t, x) \to t \cap x$. The natural base point here is the set $e = \mathbb{R}^d$.

Definition 2. Let H be a topological group (in applications, usually $\mathbb{R}$), and X a homogeneous space, for instance $\mathbb{R}$ and $\mathbb{R}^d$, or a normed vector space or a locally compact abelian group in the Bajšanski–Karamata context. We say that $k : X \to H$ is *flow homogeneous* w.r.t. the flow φ if there is a (conjugate) flow φ_k on H such that for all t, x,

$$k\big(\varphi(t, x)\big) = \varphi_k\big(t, k(x)\big).$$

Note that if k is a surjection, then the flows φ and φ_k are said to be *topologically semiconjugate* (cf. [54,61]).

Example 1 *(Multiplicative homogeneity).* For $X = \mathbb{R}^2\backslash\{(0, 0)\}$ and with $z_0 = (1, 1)$ and $s_0 = (0, 0)$, consider the standard radial flow $\varphi(t, x) = tx = (tx_1, tx_2)$. The function $k(x) = x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$ is flow homogeneous with a conjugate flow which is also radial, namely $\varphi_k(t, x) = t^\rho x$, where $\rho = (\alpha_1 + \alpha_2)$. Indeed

$$k\big(\varphi(t, x)\big) = t^{(\alpha_1+\alpha_2)} x_1^{\alpha_1} x_2^{\alpha_2} = \varphi_k\big(t, k(x)\big).$$

This is the standard notion of multiplicative homogeneity for a function F, namely

$$F(tx) = t^\rho F(x).$$

Example 2 *(Additive homogeneity).* This is significant in regular variation whenever an additive formulation is used (implying an abelian context). For $X = \mathbb{R}^2$ and with $z_0 = (1, 1)$ with s_0 at infinity, consider the shift flow $\varphi(t, x) = x + tz_0 = (x_1 + t, x_2 + t)$. The linear function $k(x) = \alpha x = \alpha_1 x_1 + \alpha_2 x_2$ is flow homogeneous with conjugate shift flow $\varphi_k(t, x) = x + (\alpha_1 + \alpha_2)t$. Indeed

$$k\big(\varphi(t, x)\big) = \alpha_1(x_1 + t) + \alpha_2(x_2 + t) = k(x) + (\alpha_1 + \alpha_2)t.$$

We note that after the standard transformation $f(x) = \log F(e^{x_1}, e^{x_2})$ additive homogeneity arising as shift-flow homogeneity for f corresponds to the standard multiplicative homogeneity for F as in Example 1.

Definition 3. (Cf. Remarks ahead of Definition 5 below.) Let H be a topological group and X a homogeneous space carrying a real flow. We say that $h : X \to H$ is *Gâteaux regularly varying* on X relative to the flow φ, with *Gâteaux limit function* $k := \partial_\varphi h$ (∂h, or $\partial_{\mathbb{R}} h$, when φ is clear from context), if for all x

$$h\big(\varphi(t, x)\big)h\big(\varphi(t, z_0)\big)^{-1} \to k(x), \quad \text{as } t \to \infty.$$

Notice that

$$k(z_0) = e_H.$$

The definition remains valid for the case of the semi-flow $(t, x) \to t \cap x$ of Example (v) above; here the limiting log conditional probability $\lim[\log P(x \cap h) - \log P(e \cap h)]$, taken over half-spaces h diverging to infinity, is studied in [2].

We think of a flow as determining a direction; to connect the real flow definition of regular variation here with the general theory one needs the following result from [12], Section 2, whose proof is by specialization to the group of actions T to $\mathbb{R}$, hence is omitted.

Proposition 1. *If h is Fréchet φ-regularly varying with respect to the group Φ, then h is Gâteaux φ-regularly varying for any Φ-flow φ.*

Proposition 2 *(Concatenation formula). If h is Gâteaux regularly varying relative to the flow φ for the distinguished point $z = z_0$, then for any w the Gâteaux limit $k_w(x) = \lim h(\varphi(t,x))h(\varphi(t,w))^{-1}$ exists and*

$$k_z(x) = k_z(w)k_w(x).$$

Proof. As before,

$$k_z(x) = \lim h\big(\varphi(t,x)\big)h\big(\varphi(t,w)\big)^{-1}h\big(\varphi(t,w)\big)h\big(\varphi(t,z)\big)^{-1} = k_w(x)k_z(w). \qquad \square$$

Thus the distinguished point has no special role, other than fixing the context. The concatenation formula shows that we can expect $k_z(.)$ typically to be affine when X and H are vector spaces, unless $k_z(0) = 0$; so since $k_z(z) = 0$, the natural choice is $z_0 = 0$. The Index Theorem below will entitle us to define the class $R_\rho(\varphi)$ with $\rho \in H$ of regularly varying functions (relative to φ) with index ρ, by analogy to the classical theory (cf. [7], Section 1.4.2); thus when $H = \mathbb{R}$, we have $\rho \in \mathbb{R}$, as in the classical theory. In fact, the result below takes in Meerschaert and Scheffler's context of work [39], namely $GL(\mathbb{R}^d)$, the group of invertible linear operators from $\mathbb{R}^d$ to $\mathbb{R}^d$, since this is an open subset of the normed vector space $H = L(\mathbb{R}^d)$ of all continuous linear operators, regarded as an additive group (their index, once transformed to the additive formulations, is in H).

On various occasions we refer to functions with properties related to the classical property of Baire. For background on Baire sets (i.e., sets with the Baire property) we refer to Kechris ([35]; see Section 8.F, p. 47) and on Baire category and *Baire spaces*, we refer to Engelking ([23]; see especially p. 198, Section 3.9 and Exercises 3.9.J), although we prefer 'meagre' to 'of first category'. In our more general context we need to distinguish between three possible interpretations of the Baire property in relation to functions, as follows.

Definitions 4.

1. Say that a function $f : X \to Y$ between two topological spaces is *$\mathcal{H}$-Baire*, for $\mathcal{H}$ a class of sets in Y, if $f^{-1}(H)$ has the Baire property (i.e. $f^{-1}(H)$ is open in X modulo the meagre sets of X) for each set H in $\mathcal{H}$. Thus f is $\mathcal{F}(Y)$-Baire if $f^{-1}(F)$ has the Baire property for all closed F in Y. Since

$$f^{-1}(Y \backslash H) = X \backslash f^{-1}(H),$$

f is $\mathcal{F}(Y)$-Baire iff it is $\mathcal{G}(Y)$-Baire, when we will simply say that f is *Baire* ('f has the Baire property' is the alternative usage). Here $\mathcal{F}(Y)$ denotes the class of closed sets and and $\mathcal{G}(Y)$ the class of open sets in Y.

2. We distinguish between functions that are $\mathcal{F}(Y)$-Baire and those that lie in the smallest family of functions closed under pointwise limits of sequences and containing the continuous functions (for a modern treatment see [34], Section 6). We follow tradition in calling these last *Baire-measurable.*

3. We call a function *Baire almost continuous*, or just *Baire-continuous*, if it is continuous when restricted to some co-meagre set.

The connections between these concepts are given in the theorems below. See the cited papers for proofs.

Banach–Neeb Theorem. *(See [3], Th. 4, p. 35, and vol. I, p. 206; [41].)*

(i) *A Baire-measurable $f : X \to Y$ with X a Baire space and Y metric is Baire-continuous.*
(ii) *A Borel-continuous $f : X \to Y$ with X, Y metric and Y separable is Baire-measurable.*

Remarks. In fact Banach shows that a Baire-measurable function is Baire-continuous on each perfect set ([3], vol. II, p. 206). Neeb assumes in addition that Y is arcwise connected, but, as Pestov remarks in [44], the arcwise connectedness may be dropped by referring to a result of Hartman and Mycielski [31] that a separable metrizable group embeds as a subgroup of an arcwise connected separable metrizable group.

Baire Continuity Theorem. *A Baire function $f : X \to Y$ is Baire-continuous in the following cases*:

(i) *Baire condition* (*see e.g.* [56], *Th.* 2.2.10, *p.* 346): *Y is a second-countable space*;
(ii) *Emeryk–Frankiewicz–Kulpa* [22]: *X is Čech-complete and Y has a base of cardinality not exceeding the continuum*;
(iii) *Pol condition* [45]: *f is Borel, X is Borelian-K and Y is metrizable and of non-measurable cardinality*;
(iv) *Hansell condition* [30]: *f is σ-discrete and Y metric.*

We will say that the pair (X, Y) *enables Baire continuity* if the spaces X, Y satisfy either of the two conditions (i) or (ii). In the applications below Y is usually the additive group of reals $\mathbb{R}$, so satisfies (i). Building on [22], Fremlin [27], Section 9, characterizes a space X such that every Baire function $f : X \to Y$ is Baire-continuous for all metric Y in the language of 'measurable spaces with negligibles'; reference there is made to disjoint families of negligible sets all of whose subfamilies have a measurable union. For a discussion of discontinuous homomorphisms, especially counterexamples on $C(X)$ with X compact (e.g. employing Stone–Čech compactifications, $X = \beta\mathbb{N}\backslash\mathbb{N}$) see [16], Section 9.

Remarks. Hansell's condition, requiring the function f to be σ-discrete, is implied by f being analytic when X is absolutely analytic (i.e. Souslin-$\mathcal{F}(X)$ in any complete metric space X into which it embeds). Kuratowski [37] first raised the general question of the circumstances when a Baire function is Baire-continuous. See Frankiewicz and Kunen [26] for set-theoretic independence results concerning Baire continuity of *all* Baire functions $f : X \to Y$ in the category of metric spaces, where such assertions are connected to certain large cardinal axioms. (See Frankiewicz [25] in regard to consequences of the axiom of constructibility.) Compare also the comments in [27], p. 86.

We will need the following result, whose proof is included as it is short and elegant (cf. Continuous Cocycle Theorem of [10], Section 5).

Banach–Mehdi Theorem. *(Cf. [3], 1.3.4, p. 40 albeit for 'Baire-measurable' functions, [40].) An additive Baire function between complete normed vector spaces is continuous, and so linear, provided the image space is separable.*

Proof. Suppose k is a Baire function, in the sense that inverse images under k of open sets are sets with the Baire property. Thus by the Baire Continuity Theorem k is continuous on some co-meagre set D. Suppose further that k is additive. If $x_n \to x_0$, then the set

$$T := \bigcap_{n\in\omega}\{t\colon\ t + x_n \in D\} = \bigcap_{n\in\omega}(D - x_n)$$

is co-meagre and so non-empty. Let $t \in T$. Thus $\{t + x_n\colon\ n \in \omega\} \subseteq D$, and so

$$k(t) + k(x_n) = k(t + x_n) \to k(t + x_0) = k(t) + k(x_0),$$

so that $k(x_n) \to k(x_0)$. Thus k is continuous. From additivity one has $k(rx) = rk(x)$, for r rational, and so from continuity for all real r. That is, k is linear. $\square$

On this matter, compare Topsøe and Hoffmann-Jørgensen ([56], Th. 2.2.12 and 2.3.1, pp. 348–350) in connection with generalizations of Banach's continuity theorem and the cautions of pp. 339–340, and in connection with the Ostrowski Theorem, see p. 368 and esp. p. 382. (See also [8] for a generalization of Banach's argument.)

We will also need to refer to a general version of the Riesz Representation Theorem (see [51], Ch. 6). For T a topological space, recall that $\mathcal{C}_c(T)$, resp. $\mathcal{C}_0(T)$, denotes the space of real-valued continuous functions which have compact support, resp. vanish at infinity. Note that a positive linear functional on $\mathcal{C}_c(T)$ need not extend to a bounded linear functional on $\mathcal{C}_0(T)$. For this reason there are two versions of the abstract Riesz representation theorem, one each for $\mathcal{C}_0(T)$ and $\mathcal{C}_c(T)$.

Theorem R1. *Let T be a locally compact Hausdorff space. For any positive linear functional k on $X = \mathcal{C}_c(T)$, there is a unique regular Borel measure μ on T such that*

$$k(x) = \int_T x(t)\,d\mu(t), \quad \text{for } x \in \mathcal{C}_c(T).$$

We also have:

Theorem R2. *Let T be a locally compact Hausdorff space. For any continuous linear functional x^* on $\mathcal{C}_0(T)$, there is a unique regular Borel signed measure μ on T such that*

$$x^*(x) = \int_T x(t)\,d\mu(t), \quad \text{for } x \in \mathcal{C}_0(T).$$

The norm of x^* as a linear functional is the total variation of μ, that is $\|x^*\| = |\mu|(T)$. Furthermore, x^* is positive if and only if the signed measure μ is non-negative.

Corollary. *Let $X = \mathcal{C}_0(T)$ with T a locally compact Hausdorff space. For Φ the group of shift homeomorphisms, if $h : X \to \mathbb{R}$ Baire Φ-regularly varying with null point the zero of X, then, for some measure μ on T we have*

$$k(x) := \partial h_\Phi(x) = \int_T x(t)\,d\mu(t).$$

In particular, if $h : \mathcal{C}[0,1] \to \mathbb{R}$ is Baire regularly varying, then, for some signed measure (function of bounded variation) α, the Φ-limit function is given by

$$k(x) := \partial_\Phi h(x) = \int_0^1 x(t)\, d\alpha(t).$$

Proof. Referring to the shift homeomorphisms $\varphi_x(z) := z + x$, we may identify x with φ_x. Since $\varphi_x \circ \varphi_y = \varphi_{x+y}$ we have by the Bounded Homomorphism Theorem ([12], Section 2)

$$k(x+y) = k(\varphi_{x+y}) = k(\varphi_x \circ \varphi_y) = k(\varphi_x) + k(\varphi_y) = k(x) + k(y),$$

since we regard $X = \mathcal{C}_0(T)$ as a (complete) normed vector space. By the Banach–Mehdi Theorem k is a continuous linear functional. The conclusion follows from the Riesz Representation Theorems R1 and R2. $\square$

3. Index Theorem

We are now ready to formulate and prove our main result, the Index Theorem of topological regular variation.

Theorem 1 *(Index Theorem). Let X, H be metrizable topological groups. The limit function $k(x) := \partial_\varphi h(x)$ of a function $h : X \to H$ which is regularly varying relative to a flow φ on X is flow homogeneous, i.e., there is a flow φ_k on H such that, for all t, x,*

$$k\big(\varphi(t,x)\big) = \varphi_k\big(t, k(x)\big).$$

Furthermore, the conjugate flow is a time-multiplicative shift in H, i.e.

$$\varphi_k(t,z) = z\rho(t),$$

where ρ is multiplicative, and continuous if the pair (X,H) enables Baire continuity. In particular, for h Baire, X and H complete normed vector spaces with H separable, there is some constant $\rho = \rho_\varphi \in H$ such that the conjugate flow takes the form

$$\varphi_k(t,z) = z + \rho t.$$

Proof. We have

$$\begin{aligned}
k\big(\varphi(\tau,x)\big) &= \lim h\big(\varphi(t+\tau,x)\big)h\big(\varphi(t,z_0)\big)^{-1} \\
&= \lim h\big(\varphi(t+\tau,x)\big)h\big(\varphi(t+\tau,z_0)\big)^{-1}h\big(\varphi(t+\tau,z_0)\big)h\big(\varphi(t,z_0)\big)^{-1} \\
&= \lim h\big(\varphi(t+\tau,x)\big)h\big(\varphi(t+\tau,z_0)\big)^{-1}h(\varphi\big(t,\varphi(\tau,z_0)\big)h\big(\varphi(t,z_0)\big)^{-1} \\
&= k(x)k\big(\varphi(\tau,z_0)\big).
\end{aligned}$$

Thus k is flow homogeneous with conjugate flow (see Definition 2)

$$\varphi_k(\tau,z) = zk\big(\varphi(\tau,z_0)\big).$$

Put

$$\rho(t) := k\big(\varphi(t,z_0)\big).$$

Since $\varphi^s\varphi^t = \varphi^{s+t}$, a reference again to the Continuous Homomorphism Theorem of [12], Section 2 yields

$$k\big(\varphi(\sigma+\tau,z_0)\big) = k\big(\varphi(\sigma,z_0)\big)k\big(\varphi(\tau,z_0)\big),$$

i.e., ρ is a homomorphism (is multiplicative), which is continuous if the pair (X,H) enables Baire continuity (see the Remark after Definitions 3). When X and H are complete normed vector spaces, regarded as additive groups, we regard the homomorphism ρ as additive. Thus if h is Baire, then so is k, and hence ρ is Baire. By the Banach–Mehdi Theorem, $\rho(t) = \rho_\varphi t$ for some ρ_φ. $\square$

Example *(Flow indices for regular variation on a normed vector space).* For X a normed vector space, the natural choice of null point is $z_0 = 0$. The canonical regularly varying functions are the continuous linear functionals in X^*, which we now investigate. Let Φ denote the family of shifts $\varphi_u(x) = x + u$, with $u \in X$. Consider an arbitrary sequence in Φ:

$$\varphi_n(x) = x + x_n.$$

Each shift is a bounded homeomorphism with $d(\varphi_n(x), id(x)) = d(x_n, 0) = \|x_n\|$. The sequence is divergent provided $\|x_n\| \to \infty$. Moreover, we have

$$x^*\big(\varphi_n(x)\big) - x^*\big(\varphi_n(0)\big) = x^*\big(\varphi_n(x) - \varphi_n(0)\big) = x^*(x).$$

Thus $h := x^*$ is $\boldsymbol{\Phi}$-regularly varying with limit function $k := x^*$ (i.e. independent of the choice of divergent sequence). We may use this observation to compute the flow index of variation ρ_φ of x^* provided the transformation group is *compatible*, that is, is contained in Φ: $\Phi(\varphi) \subseteq \Phi$, or $\varphi^t \in \Phi$, for all t. Then we have, for some $u(t) \in X$

$$\varphi^t(x) = \varphi_{u(t)}(x) = x + u(t).$$

Put $u := u(1)$ and note that the group property requires that $u(t)$ be additive, namely

$$u(s+t) = u(s) + u(t),$$

so that, since $u(t) := \varphi(t, 0)$ is continuous,

$$u(t) = tu(1),$$

and so u is the direction of the flow, since

$$\varphi^t(x) = x + tu.$$

Now

$$\rho_\varphi = \rho(1) = k\big(\varphi(1, z_0)\big) = x^*(u).$$

Remark *(Strong and weak derivatives).* The equation $\rho_\varphi = x^*(u)$ just derived identifies the index ρ_φ as a Gâteaux (directional) derivative (at infinity) of a regularly varying function in the direction of the flow u. The limit function x^* is the *Fréchet derivative* at infinity, or Φ-limit, of a regularly varying function, hence our choice of terminology (compare [32], Ch. III and [52], Ch. 10, Differentiation – omitted in 2nd ed. [53], and for more recent literature cf. [17–19]). What we see here is in complete analogy with notions of functional analysis. Fréchet differentiability is the stronger concept (most useful in optimization theory); it is usual to identify the strong derivative by computing the directional derivative as a 'function of the direction'. To motivate the following definition, recall that in the normed vector space context a function h with domain D is *Hadamard differentiable* at x 'tangentially to a subset D_0' if there is a continuous linear map (the derivative) $k(x)$ defined on D_0 such that

$$h(x + tu_t) - h(x) = tk(x)u_t + o\big(|t|\big) \quad \text{for } u_t \to u \in D_0, \text{ as } t \searrow 0;$$

see [58], Section 20.2. (The concept is used in support of the Delta method in probability and statistics; see [13,14] for examples.) The equivalent formulation is that

$$h(x + tu) - h(x) = tk(x)u + o\big(|t|\big)$$

uniformly over compact sets of points u, justifying the term *compactly differentiable* at x due to J.A. Reeds [46].

Definition 5. For X homogeneous and H a topological group, we say that $h : X \to H$ is *Hadamard φ-regularly varying*, with *Hadamard limit* function k, if for each x in X

$$h\big(\varphi(t, \sigma_t x)\big)h\big(\varphi(t, z_0)\big)^{-1} \to k(x),$$

for any bounded homeomorphisms σ_t converging to the identity. This notion occurs in course of the proof of the UCT under the guise of the crimping property (see [10]) and corresponds to a similar condition occurring in Yakimiv [60]. That same proof now gives the following result. For the notion of a Baire space see [23] (especially p. 198, Section 3.9 and Exercises 3.9.J).

Theorem 2 *(Uniform convergence theorem for flows: Flow UCT). Suppose the following*:

(i) *X is a Baire space, equipped with a flow φ;*
(ii) *X is homogeneous, i.e. for any pair of points z, u there is a bounded homeomorphism σ such that $\sigma(z) = u$.*

Let h be Baire and Hadamard φ-slowly varying: *for any bounded homemorphisms σ_t tending to the identity,*

$$h\big(\varphi(t, \sigma_t x)\big)h\big(\varphi(t, x)\big)^{-1} \to e_H.$$

Then, for x in any compact set K, we have uniformly in x the convergence as $t \to \infty$

$$h\big(\varphi(t, x)\big)h\big(\varphi(t, z_0)\big)^{-1} \to e_H.$$

This generalizes Yakimiv [60], Th. 1.1.2 (compare also [42], Cor. 1.2.8).

Example *(Flow indices for $h : \mathcal{C}[0,1] \to \mathbb{R}$).* According to the Index Theorem,

$$\rho(t) = k\big(\varphi(t, z_0)\big).$$

Thus for $z_0 = 0$ we have

$$0 = \rho(0) = k\big(\varphi(0, z_0)\big) = k(z_0).$$

We previously identified k for the shift homeomorphisms $\varphi_x(z)$ with the null point $z_0 = 0$, which ensures that $k(z_0) = 0$.

As in the last example, the compatible choice of flow is any flow of the form $\varphi(t,z) = z + tu$. Thus with s_0 at $-\infty$ we have for $u = \mathbf{1}$

$$\rho(t) := k\big(\varphi(t, z_0)\big) = \int_0^1 \big[t\mathbf{1} + z_0(s)\big]\, d\alpha(s) = t\big(\alpha(1) - \alpha(0)\big).$$

Notice that for any other distinguished point w we obtain the affine form

$$k\big(\varphi(t, w)\big) = \int_0^1 \big[t\mathbf{1} + w(s)\big]\, d\alpha(s) = t\big(\alpha(1) - \alpha(0)\big) + \int_0^1 w(s)\, d\alpha(s) = \rho(t) + k(w).$$

Remark *(Euler's Theorem).* As our notation suggests, for a fixed function h the index of regular variation (when it exists) depends on the flow. Indeed the limit function k in general depends on the flow. Consider the special case $h(x) = \alpha_1 x_1 + \alpha_2 x_2$. For the shift flow $\varphi_u(t, x) = (x_1 + u_1 t, x_2 + u_2 t)$ we obtain

$$h\big(\varphi_u(t, x)\big) - h\big(\varphi_u(t, 0)\big) = \alpha_1(x_1 + u_1 t) + \alpha_2(x_2 + u_2 t) - (\alpha_1 u_1 t + \alpha_2 u_2 t) = \alpha_1 x_1 + \alpha_2 x_2,$$

so that here $k(x) = \alpha_1 x_1 + \alpha_2 x_2$, independently of the flow u. We see that the horizontal flow $\varphi_{\mathrm{horizontal}}(t, x) = (x_1 + t, x_2)$ yields

$$k\big(\varphi_{\mathrm{horizontal}}(t, x)\big) = \alpha_1(x_1 + t) + \alpha_2 x_2 = k(x) + \alpha_1 t,$$

so that

$$\rho_{\mathrm{horizontal}} = \alpha_1.$$

Similarly, $\rho_{\mathrm{vertical}} = \alpha_2$. For the shift flow $\varphi_u(t, x) = (x_1 + u_1 t, x_2 + u_2 t)$ we of course obtain

$$\rho_u = u_1\alpha_1 + u_2\alpha_2.$$

This result is just an instance of Euler's Theorem for homogeneous functions. Indeed

$$\rho = \frac{d}{dt}\big(\rho t + k(x)\big) = \frac{d}{dt} k\big(\varphi_u(t, x)\big) = \frac{d}{dt} k(x_1 + u_1 t, x_2 + u_2 t) = u_1\alpha_1 + u_2\alpha_2.$$

This analysis may be repeated, mutatis mutandis, in the space $\mathcal{C}[0,1]$, with the vector α replaced by a function of bounded variation.

When k is differentiable and independent of the flows, as in the examples above, the Chain Rule yields, since $\rho t = k(\varphi(t, z_0))$,

$$\rho_\varphi = Dk\big(\varphi(t, z_0)\big)\dot{\varphi}(t, z_0),$$

whenever the velocity $\dot{\varphi}$ exists. This opens the issue of determining how this latter formula may be deduced from general manipulations of flows.

Consider first the case of flows $\varphi_A(t, x)$ induced by a vector field $V(x) = Ax$. We have, if A, B commute, that

$$\begin{aligned}
\rho_{A+B} &= Dk\big(\varphi_{A+B}(t, z_0)\big)\dot{\varphi}_{A+B}(t, z_0) \\
&= Dk\big(\varphi_A\big(t, \varphi_B(t, z_0)\big)\big)\dot{\varphi}_A\big(t, \varphi_B(t, z_0)\big) + Dk\big(\varphi_B\big(t, \varphi_A(t, z_0)\big)\big)\dot{\varphi}_B\big(t, \varphi_A(t, z_0)\big) \\
&= \rho_A + \rho_B.
\end{aligned}$$

In particular, for A in diagonal form: $\Lambda = \Lambda_1 + \cdots + \Lambda_n$ and $k = (\kappa_1, \ldots, \kappa_n)$ where each Λ_i has entries zero except for λ_i as the ii entry, then Λ_i describes a flow in direction e_i for which we have $\rho_i = \kappa_i\lambda_i$, so that we retrieve the Euler formula:

$$\rho_\Lambda = \kappa_i\lambda_1 + \cdots + \kappa_n\lambda_n,$$

as before. This result is at the heart of the Spectral Decomposition Theorem of [39], Cor. 2.2.5 and Th. 4.3.10.

Remark. The addition formula above holds more generally for commuting flows with velocities, as we now indicate.

Definition 6. We say that the Φ-flows φ_A, φ_B *commute* if, for all $t \in \mathbb{R}$

$$\varphi_A^t \varphi_B^t = \varphi_B^t \varphi_A^t.$$

Evidently, this will hold if the group Φ is commutative (for instance in $\mathbb{R}^d$ the group of translations $x \to x+u$). One may then show, by algebraic manipulation, first for integers and then for rationals, that

$$\varphi_A^{s+t} \varphi_B^{s+t} = \varphi_A^s \varphi_B^s \varphi_A^t \varphi_B^t,$$

i.e. for any x and rational s, t that

$$\varphi_A\big(s+t, \varphi_B(s+t, x)\big) = \varphi_A\big(s, \varphi_B\big(s, \varphi_A\big(t, \varphi_B(t, x)\big)\big)\big).$$

By continuity the identity holds for all $s, t \in \mathbb{R}$. Thus in the commuting case one may define the *sum flow* φ_{A+B} by the condition

$$\varphi_{A+B}^t := \varphi_A^t \varphi_B^t.$$

(The groups $\{\varphi_A^t\colon\ t \in \mathbb{R}\}$, $\{\varphi_B^t\colon\ t \in \mathbb{R}\}$ commute elementwise; compare the multiplicative representation of flows in [43].) In a normed vector space, assuming the flow φ_A has velocities at all points x, one has

$$\partial_t \varphi_A^t := \lim_{s\to 0}\big(\varphi_A^{t+s} - \varphi_A^t\big)/s = \lim_{s\to 0}\left(\frac{\varphi_A^s - \varphi_A^0}{s}\right)\varphi_A^t = \partial_0 \varphi_A \varphi_A^t,$$

where

$$\partial_0 \varphi_A := \lim_{s\to 0}\big(\varphi_A^s - \varphi_A^0\big)/s.$$

Thus if φ_A, φ_B commute and both have velocities at any point x, then we have

$$\partial_t \varphi_{A+B}^t = \partial_0 \varphi_{A+B} \varphi_{A+B}^t = \partial_0 \varphi_A \varphi_A^t \varphi_B^t + \partial_0 \varphi_B \varphi_B^t \varphi_A^t.$$

Note that with $\lambda\varphi(t, x) := \varphi(\lambda t, x)$, for a commutative group Φ, the Φ-flows form a vector space and if we restrict attention to flows with velocities, ∂_0 acts linearly on these.

Remarks.

1. If φ_B and φ_C commute with φ_A, then so does $\varphi_B \varphi_C$. Hence the centre of the group $\{\varphi_B\colon\ \varphi_B$ commutes with $\varphi_A\}$ is a commutative group of flows on X.

2. The centralizer of a matrix A in the group of $d \times d$ matrices coincides with the ring of polynomials in A if and only if the minimum polynomial coincides with the characteristic polynomial – see [55], 1.3 and the broader connection with spectral decomposition, also [29], Section 84.

3. A set of matrices that are normal (satisfying $AA^* = A^*A$), and so diagonalizable, commute if and only they are simultaneously diagonalizable, sharing the same orthonormal eigenvectors (see for example [38]). This is clear in the case of two matrices with distinct eigenvalues (as $Av_i = \lambda_i v_i$ implies that $A(Bv_i) = \lambda_i Bv_i$, so that $Bv_i = \mu_i v_i$ for some μ_i).

4. It seems that a parallel flow theory may be introduced. Say that the orbits, and so the flows φ_A and φ_B, are *singly intersecting* if for all x

$$O_A(x) \cap O_B(x) = \{x\},$$

or equivalently, that $\varphi_A^s x = \varphi_B^t x$ iff $s = t = 0$. Then the equation

$$\varphi_A^s \varphi_B^t z = \varphi_A^u \varphi_B^v z$$

implies for $w = \varphi_B^t z$ that

$$\varphi_A^{s-u} w = \varphi_B^{v-t} w,$$

so if the flows are singly interesting $s - u = 0 = v - t$. Thus if φ_A and φ_B commute and are singly intersecting, then the base point z defines the span of the two flows as the set of points x such that for some s, t, we have

$$x = \varphi_A^s \varphi_B^t z.$$

The co-ordinates s, t are thus uniquely determined. Moreover, by the Cocycle Theorem of [12]

$$k\big(\varphi_A^s \varphi_B^t z\big) = k\big(\varphi_A^s z\big) + k\big(\varphi_B^t z\big) = \rho_A(s) + \rho_B(t).$$

The limit functions k then have a representation theory similar to that of linear transformations on $\mathbb{R}^d$.

References

[1] B. Bajšanski, J. Karamata, Regular varying functions and the principle of equicontinuity, Publ. Ramanujan Inst. 1 (1969) 235–246.
[2] G. Balkema, P. Embrechts, High Risk Scenarios and Extremes: A Geometric Approach, EMS – European Mathematical Society Publishing House, 2007.
[3] S. Banach, Théorie des Opérations Linéaires, reprinted in: Collected Works, vol. II, PWN, Warsaw, 1979, pp. 401–411, 1st edition 1932.
[4] B. Basrak, J. Segers, Regularly varying multivariate time series, Stochastic Process. Appl. (2008), doi:10.1016/j.spa.2008.05.004, in press.
[5] A. Beck, Continuous Flows on the Plane, Grundlehren Math. Wiss., vol. 201, Springer-Verlag, 1974.
[6] Cz. Bessaga, A. Pełczyński, Selected Topics in Infinite-Dimensional Topology, PWN, 1975.
[7] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, second ed., Encyclopedia Math. Appl., vol. 27, Cambridge University Press, Cambridge, 1989, 1st edition 1987.
[8] N.H. Bingham, A.J. Ostaszewski, The converse Ostrowski theorem: Aspects of compactness, LSE-CDAM-2007-25.
[9] N.H. Bingham, A.J. Ostaszewski, Normed groups: Dichotomy and duality, LSE-CDAM-2008-10.
[10] N.H. Bingham, A.J. Ostaszewski, Topological regular variation: I. Slow variation, LSE-CDAM-2008-11.
[11] N.H. Bingham, A.J. Ostaszewski, Topological regular variation: II. The fundamental theorems, LSE-CDAM-2008-12.
[12] N.H. Bingham, A.J. Ostaszewski, Topological regular variation: III. Regular variation, LSE-CDAM-2008-13.
[13] N.H. Bingham, S.M. Pitts, Non-parametric estimation for the $M/G/\infty$ queue, Ann. Inst. Statist. Math. 51 (1) (1999) 71–97.
[14] N.H. Bingham, S.M. Pitts, Nonparametric inference from $M/G/1$ busy periods, Comm. Statist. Stochastic Models 15 (2) (1999) 247–272.
[15] L. de Haan, E. Omey, S. Resnick, Domains of attraction and regular variation in R^d, J. Multivariate Anal. 14 (1984) 17–33.
[16] H.G. Dales, Automatic continuity: A survey, Bull. London Math. Soc. 10 (2) (1978) 129–183.
[17] R.M. Dudley, Fréchet differentiability, p-variation and uniform Donsker classes, Ann. Probab. 20 (4) (1992) 1968–1982.
[18] R.M. Dudley, The order of the remainder in derivatives of composition and inverse operators for p-variation norms, Ann. Statist. 22 (1) (1994) 1–20.
[19] R.M. Dudley, R. Norvaiša, Differentiability of Six Operators on Nonsmooth Functions and p-Variation, with the collaboration of Jinghua Qian, Lecture Notes in Math., vol. 1703, Springer-Verlag, Berlin, 1999.
[20] R. Ellis, Lectures on Topological Dynamics, W.A. Benjamin, Inc., New York, 1969.
[21] R. Ellis, Cocycles in topological dynamics, Topology 17 (2) (1978) 111–130.
[22] A. Emeryk, R. Frankiewicz, W. Kulpa, On functions having the Baire property, Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (6) (1979) 489–491.
[23] R. Engelking, General Topology, Heldermann Verlag, 1989.
[24] L.R. Ford Jr., Homeomorphism groups and coset spaces, Trans. Amer. Math. Soc. 77 (1954) 490–497.
[25] R. Frankiewicz, On functions having the Baire property, II, Bull. Acad. Polon. Sci. Sér. Sci. Math. 30 (11–12) (1982).
[26] R. Frankiewicz, K. Kunen, Solution of Kuratowski's problem on function having the Baire property. I, Fund. Math. 128 (3) (1987) 171–180.
[27] D.H. Fremlin, Measure-additive coverings and measurable selectors, Dissertationes Math. (Rozprawy Mat.) 260 (1987) 1–116.
[28] W.H. Gottschalk, G.A. Hedlund, Topological Dynamics, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, RI, 1955.
[29] Paul R. Halmos, Finite-Dimensional Vector Spaces, reprinting of the 1958 second edition, Undergrad. Texts Math., Springer-Verlag, New York, Heidelberg, 1974.
[30] R.W. Hansell, Borel measurable mappings for nonseparable metric spaces, Trans. Amer. Math. Soc. 161 (1971) 145–169.
[31] S. Hartman, J. Mycielski, On the imbedding of topological groups into connected topological groups, Colloq. Math. 5 (1958) 167–169.
[32] E. Hille, R.S. Phillips, Functional Analysis and Semi-Groups, third printing of the revised edition of 1957, Amer. Math. Soc. Colloq. Publ., vol. XXXI, Amer. Math. Soc., Providence, RI, 1974.
[33] H. Hult, F. Lindskog, T. Mikosch, G. Samorodnitsky, Functional large deviations for multivariate regularly varying random walks, Ann. Appl. Probab. 15 (2005) 2651–2680.
[34] J. Jayne, C.A. Rogers, Analytic Sets, Part 1 of [50].
[35] A.S. Kechris, Classical Descriptive Set Theory, Grad. Texts in Math., vol. 156, Springer-Verlag, 1995.
[36] K. Kodaira, Über die Beziehung zwischen den Massen und den Topologien in einer Gruppe, Proc. Phys.-Math. Soc. Japan (3) 23 (1941) 67–119.
[37] K. Kuratowski, Quelques problèmes concernant les espaces métriques non-séparable, Fund. Math. 25 (1935) 534–545.
[38] M. Marcus, H. Minc, A Survey of Matrix Theory and Matrix Inequalities, reprint of the 1969 edition, Dover Publications, Inc., New York, 1992.
[39] M.S. Meerschaert, H.-P. Scheffler, Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice, Wiley, 2001.
[40] M.R. Mehdi, On convex functions, J. London Math. Soc. 39 (1964) 321–326.
[41] K.-H. Neeb, On a theorem of S. Banach, J. Lie Theory 7 (2) (1997) 293–300.
[42] E. Omey, Multivariate regular variation and applications in probability theory, Eclectica 74, EHSAL, Brussels, 1989.
[43] A.J. Ostaszewski, Regular variation, topological dynamics, and the Uniform Boundedness Theorem, LSE-CDAM-2008-15.
[44] V. Pestov, Review of [41], MathSciNet MR1473172 (98i:22003).
[45] R. Pol, Remark on the restricted Baire property in compact spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 24 (8) (1976) 599–603.
[46] J.A. Reeds, On the definition of von Mises functionals, PhD thesis, Dept. of Statistics, Harvard University, Research Report S-44, 1976.
[47] S.I. Resnick, Extreme Values, Regular Variation and Point Processes, Springer-Verlag, 1987.
[48] S.I. Resnick, Heavy-Tail Phenomena. Probabilistic and Statistical Modeling, Springer Ser. Oper. Res. Financ. Eng., Springer-Verlag, New York, 2007.
[49] S.I. Resnick, Multivariate regular variation on cones: Application to extreme values, hidden regular variation and conditioned limit laws, Cindy Greenwood Festschrift, Stochastics 80 (2008) 269–298.
[50] C.A. Rogers, J. Jayne, C. Dellacherie, F. Topsøe, J. Hoffmann-Jørgensen, D.A. Martin, A.S. Kechris, A.H. Stone, Analytic Sets, Academic Press, 1980.
[51] W. Rudin, Real and Complex Analysis, McGraw–Hill, 1966.
[52] W. Rudin, Functional Analysis, first ed., McGraw–Hill, 1973.
[53] W. Rudin, Functional Analysis, second ed., McGraw–Hill, 1991.
[54] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc. 73 (1967) 747–817.
[55] D.A. Suprunenko, R.I. Tyshkevich, Commutative Matrices, Academic Press, 1968.
[56] F. Topsøe, J. Hoffmann-Jørgensen, Analytic Spaces and Their Applications, Part 3 of [50].
[57] G.S. Ungar, On all kinds of homogeneous spaces, Trans. Amer. Math. Soc. 212 (1975) 393–400.
[58] A.W. van der Vaart, Asymptotic Statistics, Cambridge University Press, 2000.
[59] J. van Mill, Strong local homogeneity and coset spaces, Proc. Amer. Math. Soc. 133 (8) (2005) 2243–2249.
[60] A.L. Yakimiv, Probabilistic Applications of Tauberian Theorems, translated from the Russian original by Andrei V. Kolchin, Modern Probability and Statistics, VSP, Leiden, 2005.
[61] K. Yano, Topologically stable homeomorphisms of the circle, Nagoya Math. J. 79 (1980) 145–149.